JUDGEMENT AND INFERENCE

PER MARTIN-LÖF

NOTE. This is a transcript of a lecture given by Per Martin-Löf at the meeting *Inferences and proofs* at the *Université d'Aix-Marseille* on 1 June 2016. The transcript was prepared by Ansten Klev.

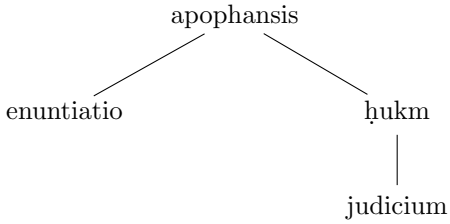
The title of this meeting we all know—*Inferences and proofs*—proofs in the sense of demonstrations, as is clear from the French version of the title, *Inférences et démonstrations*. These two terms—inference and demonstration—come in the right order here, in the sense that inference is prior—or, immediate inference, I should say—is prior to demonstration. The way I relate them is the following. I take a demonstration to be the same as a mediate inference from zero premisses. Then it remains to define mediate inference, and there the usual definition is that a mediate inference is a chain of immediate inferences from the premisses to the conclusion. That brings us back to the notion of immediate inference as the crucial notion from which to start, symbolized usually by something like this:

$$\text{(Inf)} \qquad \frac{J_1, \ldots, J_n}{J}$$

Then something is already visible, even in the choice of letter here: J is of course chosen because it can stand for judgement. This is now the reason for the title of my talk, namely *Judgement and inference*. So there is a slight polemical touch to my title. It should be compared with *Inferences and demonstrations*, and what I want to stress is that demonstrations are in the right position with respect to inferences here, but before we can deal with inferences, perhaps symbolized like (Inf), we must deal with the notion of judgement, or assertion. If you define an inference to be something like (Inf), you must first explain what are these $J_1, \ldots, J_n$, and J, and the answer is, they are assertions, or judgements.

There is then already another problem appearing here: should we say assertion, or should we say judgement? Let me very rapidly explain what the situation is. Aristotle had one notion, *apophansis*, dealt with above all in *De Interpretatione*, and it meant making a thought exterior, making it public. When it was rendered into Latin, it became *enuntiatio*, which is very fine in French, because you have *énonciation*, but it is less good in English, because, although you have enunciation, it is a dated word, it can hardly be used. That is the Boethius-translation from around 500. But then, Aristotle also went east via Syriac, so from Greek to Syriac, and then from Syriac to Arabic in the late 800s perhaps. During the Syriac phase it was just a transliteration, not a translation really, so *apophansis* became something like *apupansis*—everybody can see that it is a transliteration. But when it was rendered into Arabic, the word chosen—and it is known exactly who the translators

were—the word chosen was *ḥukm*. This is the source of the term judgement, because when the Arabic texts were translated into Latin in the 1100s ḥukm was rendered judicium. So from Aristotle's *apophansis* we got both *enuntiatio* and *judicium*.



So, two terms instead of one, and there is an interesting linguistic law, formulated by Michel Bréal, the French linguist who wrote the book *Essai de sémantique* in 1897, which is the origin of the term 'semantics', and in the second chapter of that book he deals what he calls *la loi de répartition*. It is the linguistic law which says that language does not tolerate complete synonyms for very long. What happens is that either one crystallizes out—one wins, and the other loses, and we get only one term after all—or else there is a meaning difference that develops, which makes it actually useful to have the two terms, but then the meaning has changed. It is the latter case that we are in with the terms assertion and judgement. They have both survived, because it is standard now to associate judgement with the mental act and the assertion with the verbal, exterior counterpart of the interior act of judging. That is good when you want to make this distinction between the mental and the verbal. On the other hand, it is a big insight of logic that it does not matter: the logical laws that govern our thinking are the same as the logical laws that govern our speaking. So, from that point of view, it is not good at all, because now we have lost the one term that we had before, and are more or less forced to using it either in the mental sense or in the verbal sense. If you are in that situation, what happens is that you naturally use the terms more or less interchangeably: you try perhaps to respect this mental/verbal division, but it is difficult to be systematic about it. That much about the origin of the two terms.

The basic question is, of course, What is an assertion, or judgement? I can just take over from Cesare Cozzo yesterday. I take a complete sentence to be the smallest unit of speech by means of which you can perform a speech act, that is, to really say something—that is the definition of a complete sentence, or utterance in the terminology preferred in speech act philosophy. But there are many, many different kinds of utterances in addition to assertions, of course: you have wishes and fears and commands and all this. The definition of an assertion, to my mind—I know of no better definition than to say that it is a complete sentence, or utterance, whose force is the assertoric force. I have used Frege's term here, force, in particular assertoric force. What I have in mind is that a complete utterance has, above all, to begin with, the force/content structure, which in Frege's notation is

$$\vdash C$$

so we use '$\vdash$' for the assertoric force and 'C' for the content. This is the uttermost structure of an assertion, and what makes it into an assertion is precisely that you have the assertoric force, $\vdash$, here and not some other kind of force.

Frege had a division here between two things—force and content—but it was pointed out by Hare in a paper from 1970 that, really, there are three things involved rather than two, which means that Frege's assertoric force is composed in turn out of two components. Hare chose the terminology, the rather unusual terminology: neustic, tropic, and phrastic. The phrastic part, for which a traditional term is dictum, is what I call content here, but Frege's force now has been split into neustic and tropic. Tropic is from Greek *tropos*, so *modus* in Latin, so the tropic is the mood, which in the particular case of an assertion is the assertoric mood, and the neustic stands for the madeness of the judgement. We can of course consider a judgement without making it. Frege did that by looking at $-C$. We simply consider the content, and when we actually make the judgement, Frege wrote $\vdash C$. It is the vertical stroke that Hare came to call the neustic. The important insight is that we have two things here. The mood determines if it is an assertion or a wish or a command or whatever it is, and in all cases we have to actually utter this if it is to have any effect in the conversation that we are making with other people. So the neustic you could think of as the sign, if you want, of the madeness, the performedness, of the speech act. Of course, it is not part of what is said—that is the important thing. What is said is only the tropic and phrastic—it is the saidness of what is said which is the neustic.

As you may have seen from the abstract, I want to connect this with Kantian terminology, because it is helpful to use that terminology in connection with the discussion that I want to give of so-called, in Sundholm's terminology, epistemic assumptions. Let me remind you now of Kant's modalities of judgement. Kant classified the forms of judgement into four different groups: quantity, quality, relation, and modality, each group containing three headings. I am not going to say anything about the others, but the modalities are the following:

Modality
problematic
assertoric
apodeictic

I will show how that corresponds to $\vdash C$ by the following figure.

If we begin with the content,

$$C,$$

the judgemental content, then that is not yet a judgement—it is the content of a judgement, so something has to be added to that in order for it to be a judgement at all. What has to be added is precisely the assertoric, or judgemental, mood. There is in logic no sign for just the assertoric mood. I mean, there is the assertion sign, but we have seen that the assertion sign is already composed of the neustic and the tropic, so I will use ind, because it is a received abbreviation of 'indicative' in grammar. That does not mean that it is to be taken in the grammatical sense:

it is to be taken in the logical sense, as the mood of an assertion, or judgement. If we add the assertoric mood, then what we get,

$$\mathrm{ind}\, C$$

is what Kant called a problematic judgement. The typical example would be, if we take any proposition, A, and look at

$$A \text{ true}$$
$$A \text{ false}$$

They are both problematic judgements: we just add arbitrarily a truth-ascription or a falsity-ascription to A, but we are not making the judgement—then we have two problematic judgements in Kant's sense, that is very clear from Kant's explanation of the term problematic judgement.

The term problematic is perhaps—it is difficult to be certain about what is meant to begin with, but in the idiom that we are used to, we speak about a proposition merely entertained, or merely considered was Russell's term. You can use that way of speaking and say that a problematic judgement is a judgement merely considered, or merely entertained—not made, that is.

Now suppose we are actually making the judgement, by uttering it: then we are performing the speech act of assertion, or judgement. What we get,

$$\rightarrow \mathrm{ind}\, C$$

is what Kant called an assertoric judgement. (This may already perhaps be the origin of the term assertion.)

There remains the case of an apodeictic judgement, which is yet stronger. What is added to a mere assertoric judgement in order to make it apodeictic? Well, that is the demonstration of it. If I use the usual necessity symbol, $\Box$, for apodeictic necessity—not at all for possible-worlds necessity—so apodeictic necessity, the necessity which something has because it has been demonstrated—then the whole becomes an apodeictic judgement in Kant's sense:

$$\Box \rightarrow \mathrm{ind}\, C$$

Let me now go over to what I want to reach above all, namely the discussion of the concept of epistemic assumption, which was introduced by Sundholm in a paper from 1997 and appearing also in other papers of his from approximately the same time. We are faced with an inference,

$$\frac{J_1, \ldots, J_n}{J}$$

either an entirely specific inference or a schematic inference, an inference scheme. I have chosen J because both the premisses and the conclusion are judgements, if we use the term judgement, or assertions if you use the other term. But we have seen that judgement can mean—even in Kant, it can mean at least three things: problematic or assertoric or apodeictic. It makes perfectly good sense, therefore, once we have introduced those concepts, to ask ourselves, In an inference, are the

judgements occurring as premisses—and same question for the judgement occurring as conclusion—are they problematic, assertoric or apodeictic?

Let us start with the first choice: could it be that they are problematic? That does not seem right, clearly, for basically the same reason as when we make an ordinary assumption in natural deduction, an assumption of a proposition A, say. It is clear that we are not merely entertaining, or considering, this proposition A when we make the assumption: we are actually assuming it to be true. That, I think, is clear enough. I will give no more argument for that.

It is the same if we are considering an inference or a rule of inference for its validity—we ask ourselves, Is this inference or rule of inference valid? The discussion that follows then begins invariably by saying, Assume the premisses $J_1, \ldots, J_n$—and having made those assumptions, we have to make up our mind if we have the right to judge the conclusion under those assumptions. What assumption is it that we are making? Well, I have already said that they are clearly not just problematic—we imagine ourselves in a situation where these judgements have already been made, and we are asking ourselves, Are we allowed to make J in that situation? If yes, then it is valid, and if no, then it is not valid.

So we imagine, or we assume, the judgements $J_1, \ldots, J_n$ to be made, and what is that? Well, that is precisely—they are assumed to be made, so they have the neustic, which means that they are assertoric judgements in Kant's terminology. It is therefore natural to call them assertoric assumptions. Sundholm's term is epistemic assumptions, and depending on how we understand the term epistemic, it may be in conflict or not in conflict with their being assertoric—I will say a few words about that in a moment.

Anyway, the conclusion is that the premisses are assertoric assumptions. What about the conclusion? Well, we assume the premisses to be given, and in that situation we ask ourselves if we can judge the conclusion, J. If I use this symbolic notation, the premisses, since they are assertoric, look like this:

$$\rightarrow \mathrm{ind}\, C_1, \ldots, \rightarrow \mathrm{ind}\, C_n$$

They have both the neustic, $\rightarrow$, and the tropic, ind, and the content, C_i. Now we are considering J in the conclusion. That is first of all the content C of J, but that does not make it a judgement even, we have to include the assertoric mood,

$$\mathrm{ind}\, C$$

Looking at J it appears to us in the first place as a problematic judgement, because we are asking ourselves, Do we have the right to judge J? The answer to that may be yes or no depending on whether this inference is valid or not, so J is problematic to begin with. If upon consideration we find yes, that is fine, then we assent to it,

$$\rightarrow \mathrm{ind}\, C$$

Assenting to it is the exact correspondent of Frege's judgement stroke, when we assent to a judgement.

This is my positive analysis, opting for assertoric here. What about apodeictic? That is a further possibility, that we should include the apodeictic character both in the premisses and in the conclusion. That would make the assumptions quite different, because if we include the apodeictic necessity, then the assumptions would be given in the form: assume that the judgements $J_1, \ldots, J_n$ have been demonstrated—so they should now have apodeictic force. It makes sense, of course, to assume that something has been demonstrated. One could say before the Four-colour conjecture was established, How lovely it would be if—I mean, one could say, Assume that it had been demonstrated: then we knew that we could print maps with only four colours and not just discover by chance in every specific case that we happened to be able to do so. So it makes certainly sense to assume something to be demonstrated.

But there is a strong reason why the premisses, and also the conclusion, in an inference—when we explain immediate inference, which I am in the process of doing—why they should not have apodeictic force. The reason is this. We are in the process of explaining the notion of demonstration, and we have begun by saying that a demonstration is a chain of immediate inferences. On the other hand, when validating an immediate inference, we begin by saying, Assume that the premisses $J_1, \ldots, J_n$ have been demonstrated—but then, you see, we are in a circle, because we are reducing the notion of demonstration to a chain of immediate inferences, and then it is circular to take for granted that we already know what demonstrated means in accounting for an immediate inference. So there is this circularity argument here, which has irritated me a lot for a long time, but now we have gotten something good out of it, namely that we can say, The problematic force is out, and the apodeictic force is also out because of this circularity argument that I have just given.

The outcome of this discussion is that these assumptions that Sundholm called epistemic, they should be understood as assertoric assumptions rather than apodeictic assumptions. Is that in conflict with calling them epistemic? Well, that depends on what strength you give to the term knowledge, *epistēmē*. There are two possible choices there. You could speak of knowledge in a weak sense and knowledge in a strong sense. By this distinction I mean the following. I take knowledge in the strong sense to be the, so to say, high notion, or elevated notion of knowledge, which has received so much attention in our tradition. By knowledge in the strong sense I mean demonstrated, or justified—demonstrated is a fine term in mathematics, but if you want to cover also outside mathematics, then justified is the usual term. So it is the justification that makes a judgement into knowledge in the strong sense.

The problem with giving the term knowledge only that strong sense is that it fits so badly with the way ordinary language functions. It is very strange to say that almost all knowledge that we have spent so much energy acquiring in school is not knowledge: geographical knowledge, historical knowledge and all this—certainly, we call it knowledge. It is knowledge that we have on trust, because we have been told this by our teachers or read it in our books, but we have no possibilities of

justifying this down to the bottom: we have to take these things on trust, and that is what I mean by knowledge in the weak sense.

With this terminology—knowledge in the strong, respectively weak, sense—the situation that I described in connection with (Inf) may be phrased by saying that we assume that we are allowed to trust the premisses, $J_1, \ldots, J_n$, which is to say that these epistemic assumptions constitute knowledge in the weak sense. If we agree to use the word epistemic both in the weak sense and in the strong sense, and not only in the strong sense, then there is nothing objectionable to continue to call them epistemic assumptions, but of course there is an advantage with the term assertoric assumption, namely that by changing to the term assertoric, there is no possible ambiguity any longer.

Now I am almost finished, and the reason I am finished early is that, it strikes me that I have forgotten an important point. I said that an assertion, or judgement, is a complete sentence whose mood is assertoric. I forgot to say how the assertoric mood is explained and distinguished from other moods. Of course, that is a question that speech act philosophy was designed for, to provide the explanations of the various moods. Since I am not considering any other moods than the assertoric mood, the task is easy: there is only one mood that I have to explain, but I forgot even that, so now comes, at the very end, the explanation of the assertoric mood.

If one starts from Austin—Austin really struggled with this problem, and a satisfactory formulation, or at least a somewhat satisfactory formulation, was given only by Searle in his book on speech acts. The difference with respect to Austin was that Searle classified assertion as a commissive, which is the group into which promise also is put, and promise is taken to be the first example of a commissive. People then realized that the appropriate explanation of the assertoric force was actually given much earlier by Peirce already: in 1902–03 you find it very well explained. His explanation of assertion, which I take over here, is that an assertion is the undertaking of an obligation by the speaker. The speaker is vouching for something, or guaranteeing something, or taking responsibility for something, and when he has done that, others have the right to come to him and ask him to fulfil this obligation.

This is Peirce's explanation of assertion, and that is quite striking in the sense that it brings in the notion of obligation and also, then—as I have just said, if you have made an obligation, others have the right to come to you and ask you to fulfil your obligation, and they will blame you if you cannot fulfil the obligation that you have undertaken. That means that the concepts in terms of which assertion is analyzed are the concepts of obligation and permission, or duty and right if you prefer, and they are of course deontic notions, whose place in philosophy is in normative ethics, more precisely deontological ethics. So, this analysis of the notion of assertion that was given by Peirce has some far-reaching consequences here in leading to the conclusion that the analysis of the—in the picture here,

$$\Box \to \mathrm{ind}\, C$$

where is the dividing line between the objective and the subjective? Well, it is between C and ind. The content C is what is objective, or ontological if you want, whereas all of $\Box \rightarrow \mathrm{ind}$ has to do with the act—with performing the act $(\rightarrow)$, and ind is the kind of act that we are performing. So, the dividing line between the subjective and the objective is between C and ind. The objective part, the content, that is what meaning theory is concerned with, explaining the content. But when explaining the ingredients of $\Box \rightarrow \mathrm{ind}$ we are forced back on concepts that belong to normative ethics rather, and they are precisely the concepts that are pragmatic in nature rather than objective, or ontological, in nature.

In view of this it is not strange that some logicians want that logic deals only with the objective. That succeeds by taking the basic logical notions to be proposition, truth and consequence, basically: they all belong to the objective side here. That is one possible stand to take, but then you have nothing to do at meetings like this one, which has the title *Inferences and demonstrations*, because the notions of inference and demonstration invariably involve these pragmatic concepts. Of course, the investigations of those concepts look very different from what we are used to when dealing with the objective notions in logic. That is why what we have spoken about here is a bit unusual in a sense in comparison, not with the longer tradition, but in comparison with logic as it is practised by almost all logicians presently. We are in a minority, some kind of unusual group. But I am not letting myself be turned off from dealing with these questions. On the contrary, I think it is quite interesting to see that logic does have, at bottom, a connection with practical philosophy and with normative ethics.