TRUTH OF EMPIRICAL PROPOSITIONS

PER MARTIN-LÖF

NOTE. This is a transcript of an audio recording that was made of a lecture given by Per Martin-Löf at Leiden University on 4 February 2014. The transcript was prepared by Ansten Klev.

This lecture will contain a lot of reference to Michael Dummett, and this is for the simple reason that the first time that I gave it was in connection with the *British Logic Colloquium* last September, where the first day was baptized *The Dummett Day* and was held in commemoration of Michael Dummett, since he had died relatively recently—then, one and a half year ago, and now, just over two years ago. A theme of Dummett's is the theme of how adequately to deal with truth of propositions about the past. It is a theme that he took up for the first time, to my knowledge, in a paper called "The reality of the past" from 1969 and which he struggled quite intensively with towards the end of his life, first in the Gifford Lectures from 1996–97, *Thought and Reality*, which where published only in 2006, and then in his last book, as far as I know, *Truth and the Past*, which were his Dewey Lectures in 2002, and which were published in 2004. The reason why I am meticulous about the dates here is that the Dewey Lectures, which were later, were published earlier than the Gifford Lectures, and the reason was that he changed his mind in certain respects: first he thought that the Gifford Lectures were simply superseded by the Dewey Lectures, but eventually he felt that, well, it is a reality after all that he gave the Gifford Lectures in the form that he did, and maybe it would be illuminating to compare them with the Dewey Lectures. He also has touched upon this problem in other places. One that will be of importance in my lecture today is "What is a theory of meaning? (II)", from 1976, and it is also touched upon in the early pages of *The Logical Basis of Metaphysics*, his William James Lectures from 1976.

So, this is the connection with Dummett: truth of propositions about the past. When you start thinking about this problem, you realize that it benefits from being made more general, namely to deal with truth of empirical propositions generally. This does, of course, make it more general, because if we are speaking about the past or the future, that contains a reference to the real world, for it is the real world that is conceptualized by means of the concepts of past, present and future. Truth of propositions about the past is thus part of the more general question of truth of propositions that say something about the real world, which is how I understand empirical propositions, as in the title of this talk: propositions that say something about the real world.

I put in the term empirical, and the standard term to make a contrast with empirical, from Kant, I believe, in the first place, is pure: those propositions which are not empirical are pure propositions, or ideal propositions. Some people prefer to call them eternal propositions, but then it is rather a contrast to temporal, so it is not as good, I think, as pure or ideal propositions. One might think that pure logic and mathematics is much more fancy, much more abstract, than talk about the real world, that maybe the discussion of the notion of truth of propositions should be more difficult in this abstract realm. But it is exactly the other way around: it is much simpler in this realm of pure logic and mathematics. There is something new that comes in in addition to that when you start dealing with propositions about the real world.

Let us recall now first of all: what is the explanation of the notion of proposition and truth when staying in the ideal realm of pure logic and mathematics? Of course, there are many different views on this, but the one that I will stick to in this lecture is the intuitionistic, or constructive, account of the notions of proposition and truth. On that view a proposition is defined by laying down how its canonical proofs are formed. The notion of truth is then explained by saying that in order to have the right to hold a proposition to be true, to make the judgement that the proposition is true, you should have a proof of it, a proof of it which need not be canonical, but it should at least be a method, or programme, for obtaining such a canonical proof, so that if you execute it, it ends up with a canonical proof. The termination of that execution is of course something that lies in the future, strictly speaking, so the canonical proof need not be there when the assertion is made.

This explanation that I have just given, which I hope is familiar to most of you, can be phrased more compactly by saying that the following rule is meaning determining for the form of judgement that occurs in the conclusion:

$$\frac{a : \mathrm{proof}(A)}{A \ \text{true}}$$

To say that this rule is taken as meaning determining for the form of judgement A true is just another way of phrasing what I said before. From this account of the notions of proposition and truth in the ideal world the fundamental disjunction property follows immediately, namely the property that if a disjunction is true, then either one of the disjuncts is true:

$$\begin{array}{ccc} & A \vee B \ \text{true} & \\ \swarrow & & \searrow \\ A \ \text{true} & & B \ \text{true} \end{array}$$

How does that follow? Well, we have a disjunction, so we have to lay down how the canonical proofs of the disjunction are formed, and that is by means of the introduction rules for disjunction:

$$\frac{a : A}{i(a) : A \vee B} \qquad \frac{b : B}{j(b) : A \vee B}$$

Suppose that we know a disjunction to be true. That means that we have a proof, in general non-canonical, which when executed yields a proof in one of these two forms,

$i(a)$ or $j(b)$. Then we can simply have a look whether we have i or j here. That establishes the disjunction property. This is very well known, of course, though more well known proof-theoretically than on the basis of this semantic explanation that I gave just now.

Suppose now that we leave the ideal realm and look at empirical propositions, which say something about the real world. There is an endless number of possible examples, of course, and you have a choice. If you want examples that are as humdrum as possible, then take something like the temperature at a certain place at a certain time in the past was so and so many degrees Celsius. That is good enough for the discussion of the problem that this lecture is going to deal with. But there is also the other choice of having more extravagant examples that, of course, distract from the real point, the real, so to say, technical point, or scientific point, but bring in other elements, which make them more memorable, and of a more humanistic flavour rather than of a scientific flavour. Dummett, with his classical background, no doubt favoured the second alternative. In this particular case he brought up, in "What is a theory of meaning? (II)", the following couple of examples.

> There is now either an odd or an even number of ducks on the pond.

Here, I guess, we should imagine him in Oxford, where there is a pond somewhere, and he might be in a room immediately overlooking the pond, or at least in such a circumstance that he could very easily and rapidly go out and count the number of ducks in the pond, if need there be, and tell whether it was odd or even. This example, as you see, is in the present tense: there *is* now either an odd or an even number of ducks on the pond. The example that he couples with it is

> There was either an odd or an even number of geese which cackled on the Capitol.

This is the example that I will deal with at length today.

I must confess that when I first saw this example, which must have been soon after 1976, I thought—stupidly—that presumably this was something Dummmett thought up, at random, so to say, something in the past that we could not possibly know about in the present, because all traces have been lost. Surely, that was not the case. He is referring to a very particular event on the 18th of July in 390 BC, although nowadays the year is sometimes corrected to 387 (I have absolutely no idea why). The story that Dummett is presupposing here was told by Livy. It has to do with the attack on Rome by the Gauls in this year, and this time of the year. The Romans were defeated except that they still held the hill of the Capitol, and at night the people from the Gauls climbed the hill, and it was successful—although a very difficult thing, to climb the hill—but when they reached the top of the hill the geese which were kept there, sacred to Juno, the geese cackled, and hence Marcus Manlius, who was the commander on the Roman side, and who got the epithet Capitolinus precisely because of this event, woke up and was able to fight the attacking Gauls down. This is, no doubt, the event that Dummett had in mind with this example. Of course, these details are completely irrelevant for the logical problems that we are discussing, but, no doubt, when you have heard this story, you will remember this example more easily. Dummett definitely had a taste for examples of this more extravagant kind.

Now we have a particular example, I am thinking of the second one, which is a disjunction: there was either an even or an odd number of geese which cackled on the Capitol. You see that the disjunction property fails here, because we have no possibility, as we have in the ideal case, of deciding—I take it that we all agree that this disjuntion is clearly true: we do not know which particular number it was, but any number is either even or odd, so whatever it was, it was either even or odd. I take that to be unproblematic. So, the disjunction is clearly true, but we have no means of deciding whether the geese were even or whether they were odd in number, and this is because no knowledge of that kind has been handed down to us by the tradition. The whole story was handed down to us in the first place by Livy, as I said, and of course, in that story there might have been information about the number of geese being 12 or something, but there was not, and we have no other sources either. The problem is thus not that we do not know how we should have to go about deciding this disjunction: we should have had to be there on the the Capitol at the time in question to count them. The number of geese is the number that we should have obtained if we were to have been there to count them. Here, for the first time in my talk, the counterfactual character comes in, I mean, what we nowadays in philosophy call counterfactuals, from the 1950s perhaps, but which in traditional grammar were called subjunctive conditionals, which is the irreal case, the *casus irrealis*, of a conditional, the irreal case referring to the fact that the antecedent of the conditional is false, counter to how things really are.

That is therefore what we should have had to do in order to decide this disjunction, but it is no longer possible, hence we are unable to reach this state of knowledge,

> There was an odd number of geese which cackled on the Capitol,

or this state of knowledge,

> There was an even number of geese which cackled on the Capitol.

That shows immediately that the account of truth for ideal propositions breaks down in the empirical case. The challenge, then, for a logician is to provide the appropriate formalization, the formal representation of this example, which brings out the logic behind it. Strangely enough, I would say, this has not been done. These examples have been well known for decade after decade after decade, but no-one sat down and tried simply to do it, to formalize it properly. This is what has given rise to the choice of this topic for my talk.

First of all, we have to see how the notion of proposition is generalized. We no longer have only ideal propositions, explained in the way I stated in the beginning: we now also have empirical propositions. So we need to define what an empirical proposition is. In doing so, one has to start by explaining the concept of an empirical quantity. What is an empirical quantity? As always, examples is the first step to explaining what something is. What are examples of empirical quantities? Well, it could be my sex, for instance, or my eye colour, or my height, or something like that, typical characteristics of humans, or it could be the length of this table, or the breadth of this table, or it could be the number of geese on the Capitol this particular night in 390 BC: that is an empirical quantity. In these examples the quantities that I have chosen have different types, clearly. The sex is an element of the two-element set of male and female. For my eye colour, there are more alternatives, brown and blue and a few more perhaps, but it is another set than the two-element set. My height is normally given in centimeters, so if you take all the natural numbers, it is big enough at least—it would be sufficient with up to 300 or something, but it is more convenient, usually, to take all the numbers. In the example with the geese, it is again a number. So, these empirical quantities are typed.

So far I have just given examples. To say something in general about what an empirical quantity is, my attempt is to say that a purely mathematical quantity—that is, the contrasting kind of quantity—is one whose value is determined by calculation, whereas an empirical quantity is one whose value is determined instead by experiment. The experiment in the above examples—it is a rather general use of the term experiment—is the experiment of simply taking a look at my eyes to tell what colour they are, or using the usual instrument to measure my height, and similarly in the other cases. Because of the example with the geese and the Capitol, I also count counting to be an experiment: if you have a certain number of apples, and you count the apples, then I consider that an experiment that yields as result the number of apples. You thus see there is a contrast here between calculation and experiment: calculation for pure quantities and experiment for empirical quantities.

One philosopher who has devoted a lot of reflection to the relation between calculation and experiment is Wittgenstein. In the *Tractatus* there is the statement that calculation is not an experiment. He also says in one place in the *Philosophical Investigations* that a calculation is not an experiment, and then there is much more about this in the *Remarks on the Foundations of Mathematics*. There, he is less categorical, it seems to me, though on the other hand, it is quite difficult to find out exactly what his view was, whether calculation was or was not an experiment.

I would say myself that they are different, but they stand in a certain relation to each other. They are analogous in some way, and the analogy is the same as that between digital and analogue computation. Analogue computation, as with a slide ruler or some physical device of that kind, is an experiment: you can use a physical experiment instead of a calculation to find out the result of a calculation. Still, I think we have to uphold the difference between calculation and experiment.

Let us look at differences and similarities here. In the case of a calculation, we pass from 2^3 to 8, or something, but in the case of an empirical quantity—and I will use a notation like this,

$$X$$

for an empirical quantity—the steps that we go through are the steps in the carrying out of the experiment, and then we get some result, that my eye colour is brown, or whatever it is. The result of a calculation is usually reported in this way,

$$2^3 = 8$$

and we report the result of an experiment in a similar way,

$$X = \text{brown}$$

and speak of brown as the value of X, just as in the calculation, we speak of 8 as the value, or result, of the computation. So, here we have obvious similarities. The difference is that the whole passage from 2^3 to 8 goes on in pure mathematics and pure logic, stepwise following the definitions, whereas the passage from X to brown does not go on in pure mathematics: it goes on in the real world, where we carry out this experiment and observe the result.

Now I have already started to use a symbol for an empirical quantity. We need some convenient symbol for empirical quantities, and there we are in the fortunate situation that the probabilists and statisticians have already done that for us, because, surely, their whole subject is about empirical quantities of the particular kind that are called random quantities, whose value is determined by a random experiment, like coin tossing or tossing a die, or something like that. That is an example of a particular kind of empirical quantity, so we can just adopt the notation that is well established there for empirical quantities in general, and one standard notation is to use capital letters of the kind that you have at the end of the alphabet,

$$X, Y, Z$$

From that point of view, they are like variables, they are from the end of the alphabet, but, they are now capital instead of small, which is usual for ordinary variables. This notation, which is used very efficiently—it is a brilliant invention, really, in probability theory—this particular choice is, as far as I know, due to Paul Lévy, in his *Calcul des probabilités* from 1925, but is now adopted generally. This is the notation that I shall adopt in this lecture for empirical quantities in general. Let this be enough about what empirical quantities are.

Now I can explain what an empirical proposition is. By definition an empirical proposition is a propositional function in the ordinary sense of certain empirical quantities, which is to say that an empirical proposition is something of the form

$$A(X_1, \ldots, X_n) \tag{1}$$

where A is an ordinary propositional function:

$$A(x_1, \ldots, x_n) \text{ prop} \quad (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1})) \tag{2}$$

This is an ordinary ideal propositional function, and it belongs to pure logic and mathematics, but now we are inserting into its argument places certain empirical quantities:

$$(3) \qquad \begin{aligned} &X_1 : A_1 \\ &\quad\vdots \\ &X_n : A_n(X_1, \ldots, X_{n-1}) \end{aligned}$$

This means that, whereas (2) is open, depending as it does on variables, (1) is not open any longer, because empirical quantities are not variables in the usual sense. That is clear, I suppose: my height, for instance, is not a variable, it is something which has a definite value, which you determine by making this experiment. Hence, a proposition which says something about my height, that it is more than 1.50m, or something like that, that is a closed, or definite, proposition, not open, depending on a variable.

If we go over to the values of the empirical quantities, which are determined by experiment, then those values, as you see—the height is a natural number, brown is not such a common object of pure mathematics, so let us code the blue and brown eye colour by 0 and 1 instead—the types that we have here come from pure mathematics, and the values similarly belong to pure mathematics, once we have made a representation, as I did in this case, of the colours by some code that we have available in pure mathematics. The values are, say

$$(4) \qquad \begin{aligned} &X_1 = a_1 : A_1 \\ &\quad\vdots \\ &X_n = a_n : A_n(a_1, \ldots, a_{n-1}) \end{aligned}$$

Here we have an assignment of values to the empirical quantities, and they should be obtained by performing the appropriate experiment, the experiment which defines the empirical quantity in question. You saw how I gave examples by telling, Do this—that explains the notion of height, and so on.

Now we have something here which in one way looks awfully familiar, and in another way looks new. For what is (4) if not a protocol in the sense that one has spoken about protocol sentences in logical empiricism in the early 1930s? This definitely deserves to be called a protocol, which contains all the experimental data that we have acquired. We see something new at the same time, namely that protocol sentences—Protokollsätze as they were dealt with in the 1930s—were taken to be atomic propositions in the sense of predicate logic, whereas here what comes out on this analysis are not propositions, but rather assertions or judgements. So we should speak of protocol judgements, rather than protocol—ja, there is the problem with the word Satz in German: Protokollsätze, that is unproblematic, but when you translate it into English, you make it into sentences or propositions, and I do not want to be involved in that. However you translate it, they are not judgements, because there is a clear distinction between propositions and judgements. So, as I have said, they are really familiar, being the protocol of the experiment, but they are not protocol Sätze, they are protocol judgements.

To compare now with probability theory, (4) is what is called a sample point of the sample space in probability theory, while (3) corresponds to what is called the sample space, that is, all the possible values that these empirical quantities may obtain. The sample space is standardly denoted by Ω in probability theory, and a particular point of the sample space, as in (4), is typically called ω, which is an element of Ω. It is of course not necessary for me to introduce this probabilistic terminology, you can just take it as it is, but I want to acknowledge the fact that I am doing something here which is very well known to probabilists and statisticians.

Now we have reached the notion of an empirical proposition, and we can pass over to the notion of truth of empirical propositions. What does it mean for an empirical proposition to be true? The first attempt to answer it would be to try to reduce the problem to what we already know, namely the answer for ideal propositions, by saying that what it means for the empirical proposition (1) to be true is that

$$A(a_1, \ldots, a_n) \tag{5}$$

is true, where a_1 is the value of the empirical quantity X_1 and a_n the value of X_n. This is such a sensible answer that there could hardly be anything wrong with it, but on the other hand, it needs to be deciphered further. What we have in (5) is an ideal proposition, because the values of the empirical quantities are pure mathematical quantities, as I have already said, and A is an ideal propositional function. So (5) is a wholly ideal proposition, and what it means for it to be true is unproblematic for us. The more difficult point is, What does it mean to say that $a_1, \ldots, a_n$ are the values of $X_1, \ldots, X_n$ respectively? That is clear enough if these values have been determined and are available to us: then we just insert them as arguments in $A(X_1, \ldots, X_n)$ and ask whether this is true or not. But the problem is that we are now dealing with empirical quantities like the number of geese that cackled on the Capitol in 390 BC, and what is that? Well, that is—I take it without argument—the same as the number that we should have observed if we were to have counted the geese on the Capitol that night. So, it is a counterfactual term. If X_1 would be the number of geese, then the a_1 here is a counterfactual term, the number of geese that we should have arrived at if we were to have counted them. That is a quite complicated notion, which we have to analyze properly, logically.

In analyzing the notion of truth of ideal propositions, that is, the simple case, a good strategy is to bring in the subject, bring in the subjectivity, or the knowing subject, and ask instead, Under what conditions do I have the right to judge a proposition to be true? And the answer was, you should have a proof, not necessarily canonical, of it. Let us adopt the same strategy in this more complicated case. Under what conditions are we entitled to judge an empirical proposition to be true when we are not in the simple situation that we know the values of all the empirical quantities? If we know their values already, then we are in the case of (5), which is simple. But in general, we do not have complete information about the values of the empirical quantities, as in the geese example.

My answer to the question, Under what conditions are we entitled to judge an empirical proposition to be true?, is that we are entitled to judge it to be true provided—let me introduce the notation first. We may have the values of some of these quantities, but not of all of them—that is what causes the difficulty. Let us say that we have the values of the first m of the n quantities,

$$X_1 = a_1 : A_1 \\ \vdots \\ X_m = a_m : A_m(a_1, \dots, a_{m-1}) \tag{6}$$

and the last ones are unknown to us. Then we may judge the empirical proposition to be true provided we may judge

$$A(a_1, \dots, a_m, x_{m+1}, \dots, x_n)$$

to be true in the context

$$x_{m+1} : A_{m+1}(a_1, \dots, a_m), \dots, x_n : A_n(a_1, \dots, a_m, x_{m+1}, \dots, x_{n-1})$$

We insert the values of the quantities whose values are known, just as before, but the remaining quantities, whose values are not known, we simply replace by variables. This is, so to say, the best we can do if we do not know anything about what the value is: we insert some arbitrary x.

I have now given the condition for us to have the right to judge an empirical proposition to be true in the case when we do not know the value of all the empirical quantities, only of some of them. You see here the good notational difference, introduced in probability theory, between capital letters for the random quantities and small letters for ordinary variables.

So, this is the formal analysis. Let us go back to Dummett's example. I will take as my second example the sea fight, chapter nine of Aristotle's *De Interpretatione*, but to begin with, Dummett's example. We have one empirical quantity,

X = the number of geese that cackled on the Capitol
= the number which we should have obtained if were to have counted them

Its type is the type of natural numbers,

$$X : N$$

The Realism–Idealism problem comes in already here, and that is why Dummett has been so engaged by this problem. In his terms, at least, the Realist view is that X denotes a determinate natural number, because there must have been a determinate number there at that time, because the geese flock is not big, and there is no problem of talking about the number of geese in a geese flock of ordinary size, domestic size, let us say. So, the Realist view is no doubt that X denotes a determinate number—which is unknown to us, but that is another matter.

Since X is not known to us, since it has this counterfactual form, it cannot be interpreted as in (4), because these were the known quantities, and we do not know it, so we have to interpret it as a variable instead:

$$X = x : N \quad (x : N)$$

This is the interpretation of X. In probability theory one is familiar with the fact that random quantities are interpreted as functions over the underlying sample space, and that is precisely what happens here: it is the identity function on the underlying sample space N.

Then we can return to the disjunction property. The formalization of Dummett's proposition now, there was either an even or an odd number of geese that cackled on the Capitol, is

$$\text{odd}(X) \vee \text{even}(X) \text{ true} \tag{7}$$

where odd is the usual predicate "there is an n such that $x = 2n + 1$" and even is "there is an n such that $x = 2n$'. By definition, this amounts to the judgement

$$\text{odd}(x) \vee \text{even}(x) \text{ true} \quad (x : N)$$

That is simply how I explained the notion of truth of an empirical proposition. Compare this to the disjunctive alternatives:

$$\text{odd}(X) \text{ true} \qquad \text{even}(X) \text{ true} \tag{8}$$

They get interpreted under this interpretation as

$$\text{odd}(x) \text{ true} \quad (x : N) \qquad\qquad \text{even}(x) \text{ true} \quad (x : N)$$

On this interpretation, therefore, (7) comes out correct, clearly, but equally clearly, neither of the judgements in (8) is correct. Hence the disjunction property fails because of the counterfactual character of the term "the number which we should have obtained if were to have counted them". That counterfactual term gets interpreted as a function over an underlying sample space, and because of that, (7) can be correct, although both of the judgements in (8) are incorrect, unlike in the case of ideal propositions.

Let me give the final example, of the sea fight. There we have a very typical case again of what I said before that one can choose a humdrum example and one can choose a more fancy example. The humdrum example nowadays, which has all the logic in it, and hence is better from a certain point of view, in that nothing distracts you from this logic, is coin tossing. I have a coin here, and I can toss it, and it turns out either heads or tails. But in Aristotle, who has the first discussion in our tradition of the peculiarities of this situation, it is not coin tossing, but it is whether there will be or there will not be a sea fight tomorrow. In that case, the previous example has to be only slightly changed.

I introduce an empirical quantity,

$$X = \begin{cases} 1 & \text{if there is a sea fight} \\ 0 & \text{if there is no sea fight} \end{cases}$$

Then X is an element of the two-element set, called bool nowadays in computer science,

$$X : \text{bool}$$

The judgement that there will be a sea fight tomorrow or there will not be a sea fight tomorrow is then

$$I(\text{bool}, X, 1) \vee I(\text{bool}, X, 0) \text{ true} \tag{9}$$

The big difference between this example and Dummett's example is that now we are dealing with the future, whereas in Dummett's example we were dealing with the past. The procedure is the same, but X is now unknown because it is the future, so we could not possibly know it, whereas in the other case, we might have known it, but we actually do not. In any event, we should replace X by a variable,

(10) $$I(\text{bool}, x, 1) \vee I(\text{bool}, x, 0) \text{ true} \quad (x : \text{bool})$$

The two alternatives are

(11) $$I(\text{bool}, X, 1) \text{ true} \qquad I(\text{bool}, X, 0) \text{ true}$$

When we decipher those, we get

(12) $$I(\text{bool}, x, 1)\,\text{true}\;\;(x : \text{bool}) \qquad I(\text{bool}, X, 0)\,\text{true}\;\;(x : \text{bool})$$

The judgement (10) is unproblematically correct: every boolean value is either equal to 1 or to 0. But both judgements in (12) are obviously incorrect: it is not true that any boolean value is equal to 1, and it is not true that any boolean value is equal to 0. In this respect, the situation is completely analogous to Dummett's example.

The way Aristotle expressed this conclusion was to say that it is necessarily the case that there either will be or there will not be a sea fight tomorrow, but it is not necessary that there will be a sea fight tomorrow nor that there will not be a sea fight tomorrow. The necessity in Aristotle's way of expressing it is the generality that you have over the context. The disjunction in (9) is necessarily true, but neither disjunct is necessarily true, so neither of the judgements in (11) is necessary. This was Aristotle's way of expressing it, which is of course also the way that we have in modern modal logic. For some strange reason we do not express ourselves that way in probability theory, and I think that is basically owing to the fact that when modern probability theory was developed, around 1920 say, there was no tie with these older things, with modal logic, the sea fight, and such things, so one simply said that an event is certain if it covers the whole sample space—so, certain was the term one introduced, sicher, rather than necessary, but had one known more about the tradition, then one would, no doubt, have chosen necessary instead.

With this I have reached the end, and maybe—we do not know—maybe Dummett will have succeeded in having introduced an example that will be remembered in the future.